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Forward–backward stochastic differential equations with nonsmooth coefficients

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Abstract

Solvability of forward–backward stochastic differential equations with nonsmooth coefficients is considered using the Four-Step Scheme and some approximation arguments. For the one-dimensional case, the existence of an adapted solution is established for the equation which allows the diffusion in the forward equation to be degenerate. As a byproduct, we obtain the existence of a viscosity solution to a one-dimensional nonsmooth degenerate quasilinear parabolic partial differential equation. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

In this paper, we consider the solvability of following stochastic differential equation:

$$\begin{aligned} &\mathrm{d}X(t)=b(t,X(t),Y(t))\,\mathrm{d}t+\sigma(t,X(t),Y(t))\,\mathrm{d}W(t), \quad t\in[0,T],\\ &\mathrm{d}Y(t)=h(t,X(t),Y(t),Z(t))\,\mathrm{d}t+Z(t)\,\mathrm{d}W(t), \quad \forall t\in[0,T],\\ &X(0)=x, \quad Y(T)=g(X(T)), \end{aligned} \tag{1.1}$$

on some given filtered probability space $(\Omega,\mathscr{F},\{\mathscr{F}_t\}_{t\geqslant 0},\mathbf{P})$, with b,σ,h,g being some given maps. Here, $W(\cdot)$ is a standard n-dimensional Brownian motion defined on this probability space, whose natural filtration is $\{\mathscr{F}_t\}_{t\geqslant 0}$. The unknown processes that we are looking for are $(X(\cdot),Y(\cdot),Z(\cdot))$, which takes values in $\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^{m\times n}$. The key requirement is that this triple has to be $\{\mathscr{F}_t\}_{t\geqslant 0}$-adapted. Such an equation has been

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called a forward–backward stochastic differential equation (FBSDE). The study of such equations was started in the early 1990s. Quite a few authors have contributed to the theory of FBSDEs. Among them, we would like to mention Antonelli (1993), Ma et al. (1994), Hu and Peng (1995), Ma and Yong (1995, 1999b), Yong (1997, 1999a), Pardoux and Tang (1999) and Hu (1999b,c). See Duffie et al. (1995), Cvitanic and Ma (1996), Buckdahn and Hu (1998a,b), Hu (1999a) and Briand and Hu (1999) for some other related works, and see the recent monograph by Ma and Yong (1999a) for a systematic survey.

Eq. (1.1) can be regarded as an extension of the so-called backward stochastic differential equation (BSDE) of the following form:

$$\begin{aligned} &\mathrm{d}Y(t) = h(t, Y(t), Z(t))\,\mathrm{d}t + Z(t)\,\mathrm{d}W(t), \quad t \in [0, T], \\ &Y(T) = \xi. \end{aligned} \tag{1.2}$$

When $h(t, y, z)$ is linear in (y, z), such an equation was first studied by Bismut (1976) in the context of maximum principle for stochastic optimal controls. The general nonlinear case of (1.2) was first investigated by Pardoux and Peng (1990). See also Duffie and Epstein (1992) in a different formulation. Later developments of BSDEs can be found in El Karoui et al. (1997) (see Yong and Zhou, 1999, also).

In Ma et al. (1994), an approach called Four-Step Scheme was introduced. Under certain nondegenerate and smoothness conditions, it was proved that (1.1) admits a unique adapted solution. On the other hand, in Ma and Yong (1999b), under less smoothness conditions, and without assuming the nondegeneracy condition, the approximate solvability has been introduced and studied for FBSDEs. In Hu (1999b), a result of the existence of a solution to the one-dimensional FBSDE (i.e., $n = m = 1$) with nonsmooth coefficients is established under the nondegeneracy condition. The purpose of this paper is to prove the solvability (not just approximate solvability) of FBSDE (1.1) with nonsmooth coefficients, still under the nondegeneracy condition, in the case of higher dimensions. In addition, if the dimension is one, i.e., $n = m = 1$, we even relax the nondegeneracy of the forward diffusion, which leads to the existence of viscosity solution to a degenerate nonsmooth quasilinear parabolic PDE. Such a result seems to appear here at the first time, to our best knowledge.

The rest of the paper is organized as follows. In Section 2, we recall the so-called Four-Step Scheme. Section 3 is devoted to the approximation and a proof of our main result. In Section 4, we study the one-dimensional case.

2. Assumptions and Four-Step Scheme revisited

In this section, we will recall the so-called four-step scheme and will list our assumptions.

First of all, we make the following basic assumption.

(H1) $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t \geqslant 0}, \mathbf{P})$ is a filtered probability space satisfying the usual condition (Ma and Yong, 1999; Yong and Zhou, 1999), on which an n-dimensional standard Brownian motion $W(\cdot)$ is defined such that $\{\mathscr{F}_t\}_{t \geqslant 0}$ is the natural filtration of $W(\cdot)$.

(H2) The functions b: $[0,T]\times\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^n$, σ: $[0,T]\times\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^{n\times n}$, h: $[0,T]\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^{m\times n}\to\mathbb{R}^m$ and g: $\mathbb{R}^n\to\mathbb{R}^m$ are uniformly continuous and satisfy the following Lipschitz condition: There exists a constant $L>0$ such that (for $\varphi=b,\sigma,h,g$)

$$
\begin{aligned}
&|\varphi(t,x,y,z)-\varphi(t,\bar{x},\bar{y},\bar{z})|\leqslant L(|x-\bar{x}|+|y-\bar{y}|+|z-\bar{z}|),\\
&\quad\forall t\in[0,T],\ x,\bar{x}\in\mathbb{R}^n,\ y,\bar{y}\in\mathbb{R}^m,\ z,\bar{z}\in\mathbb{R}^{m\times n},
\end{aligned}
\tag{2.1}
$$

and moreover, g is bounded.

Let $L^2_{\mathscr{F}}(0,T;\mathbb{R}^n)$ be the set of all $\mathbb{R}^n$-valued $\{\mathscr{F}_t\}_{t\geqslant 0}$-adapted processes $\varphi(\cdot)$ such that $E\int_0^T|\varphi(t)|^2\,\mathrm{d}t<\infty$, and let $L^2(\Omega;C_{\mathscr{F}}([0,T];\mathbb{R}^n))$ be the set of all continuous $\mathbb{R}^n$-valued $\{\mathscr{F}_t\}_{t\geqslant 0}$-adapted processes $\varphi(\cdot)$ such that $E[\sup_{t\in[0,T]}|\varphi(t)|^2]<\infty$. Also, let $C^{1,2}([0,T]\times\mathbb{R}^n;\mathbb{R}^m)$ be the set of all $\mathbb{R}^m$-valued functions $u(t,x)$ which is C^1 in t and C^2 in x. The following definition is necessary.

Definition 2.1. A triple of processes $(X(\cdot),Y(\cdot),Z(\cdot))$ with $X(\cdot)\in L^2(\Omega;C_{\mathscr{F}}([0,T];\mathbb{R}^n))$, $Y(\cdot)\in L^2(\Omega;C_{\mathscr{F}}([0,T];\mathbb{R}^m))$, $Z(\cdot)\in L^2_{\mathscr{F}}(0,T;\mathbb{R}^{m\times n})$ is called an adapted solution of (1.1) if the following holds almost surely:

$$
\left.
\begin{aligned}
X(t)&=x+\int_0^t b(s,X(s),Y(s))\,\mathrm{d}s+\int_0^t\sigma(s,X(s),Y(s))\,\mathrm{d}W(s),\\
Y(t)&=g(X(T))-\int_t^T h(s,X(s),Y(s),Z(s))\,\mathrm{d}s-\int_t^T Z(s)\,\mathrm{d}W(s),
\end{aligned}
\right\}\quad t\in[0,T].
\tag{2.2}
$$

Let us now recall the Four-Step-Scheme (Ma et al., 1994). Suppose $(X(\cdot),Y(\cdot),Z(\cdot))$ is an adapted solution to (1.1). We assume that $Y(\cdot)$ and $X(\cdot)$ are related by

$$
Y(t)=u(t,X(t)),\quad\forall t\in[0,T]\quad\text{a.s.-}\mathbf{P}.,
\tag{2.3}
$$

where u is some function to be determined. Let us assume that $u\in C^{1,2}([0,T]\times\mathbb{R}^n;\mathbb{R}^m)$. Then by Itô's formula, we have, for $1\leqslant k\leqslant m$

$$
\begin{aligned}
\mathrm{d}Y^k(t)&=\mathrm{d}u^k(t,X(t))\\
&=\{u^k_t(t,X(t))+\langle u^k_x(t,X(t)),b(t,X(t),u(t,X(t)))\rangle\\
&\quad+\tfrac{1}{2}\mathrm{tr}\,[u^k_{xx}(t,X(t))(\sigma\sigma^{\mathrm{T}})(t,X(t),u(t,X(t)))]\}\,\mathrm{d}t\\
&\quad+\langle u^k_x(t,X(t)),\sigma(t,X(t),u(t,X(t)))\,\mathrm{d}W(t)\rangle.
\end{aligned}
\tag{2.4}
$$

Comparing (2.4) with (1.1), we see that if u is the right choice, it should be that, for $k=1,\dots,m$,

$$
\begin{aligned}
&h^k(t,X(t),u(t,X(t)),Z(t))=u^k_t(t,X(t))+\langle u^k_x(t,X(t)),b(t,X(t),u(t,X(t)))\rangle\\
&\quad+\tfrac{1}{2}\mathrm{tr}\,[u^k_{xx}(t,X(t))(\sigma\sigma^{\mathrm{T}})(t,X(t),u(t,X(t)))],\\
&u(T,X(T))=g(X(T)),
\end{aligned}
\tag{2.5}
$$

and

$$u_x(t,X(t))\sigma(t,X(t),u(t,X(t)))=Z(t). \tag{2.6}$$

The above heuristic arguments suggest the following *Four-Step Scheme* for solving the FBSDE (1.1) (see Ma et al., 1994).

The Four Step Scheme:

Step 1: Set: (In the current case, this step can be omitted.)

$$z(t,x,y,p)=p\sigma(t,x,y), \quad \forall (t,x,y,p)\in[0,T]\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^{m\times n}. \tag{2.7}$$

Step 2: Solve the following parabolic system for $u(t,x)$:

$$\begin{aligned} &u_t^k+\tfrac{1}{2}\mathrm{tr}\,[u_{xx}^k(\sigma\sigma^{\mathrm{T}})(t,x,u)]+\langle b(t,x,u),u_x^k\rangle \\ &\qquad -h^k(t,x,u,u_x\sigma(t,x,u))=0, \quad (t,x)\in[0,T]\times\mathbb{R}^n, \quad 1\leqslant k\leqslant m, \\ &u(T,x)=g(x), \quad x\in\mathbb{R}^n. \end{aligned} \tag{2.8}$$

Step 3: Using u obtained in Step 2 to solve the following forward SDE:

$$\begin{aligned} &\mathrm{d}X(t)=b(t,X(t),u(t,X(t)))\,\mathrm{d}t+\sigma(t,X(t),u(t,X(t)))\,\mathrm{d}W(t), \quad t\in[0,T], \\ &X(0)=x. \end{aligned} \tag{2.9}$$

Step 4: Set

$$\begin{aligned} &Y(t)=u(t,X(t)), \\ &Z(t)=u_x(t,X(t))\sigma(t,X(t),u(t,X(t))). \end{aligned} \tag{2.10}$$

If the above scheme is realizable, (X,Y,Z) would give an adapted solution of (1.1). In the proof, the following result, which can be proved by Theorem 7.1 of Ladyzenskaja et al. (1967, pp. 596–597, Chapter VII), plays an essential role.

Lemma 2.2. *Suppose that all the functions σ, b, h are C^2 in all its arguments and $g\in C^{2+\alpha}(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$ (for some $\alpha\in(0,1)$). Suppose also that for all $(t,x,y,p)\in[0,T]\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^{m\times n}$, it holds that*

$$v(|y|)I\leqslant(\sigma\sigma^{\mathrm{T}})(t,x,y)\leqslant\mu(|y|)I, \tag{2.11}$$

$$|b(t,x,y)|\leqslant\tilde{\mu}(|y|), \tag{2.12}$$

$$|(\sigma\sigma^{\mathrm{T}})_{x_i}(t,x,y)|+|(\sigma\sigma^{\mathrm{T}})_{y_k}(t,x,y)|\leqslant\tilde{\mu}(|y|), \quad 1\leqslant i\leqslant n, \ 1\leqslant k\leqslant m \tag{2.13}$$

for some continuous functions $\tilde{\mu}(\cdot)$, $\mu(\cdot)$, and $v(\cdot)$, with $v(r)>0$;

$$|h(t,x,y,p\sigma(t,x,y))|\leqslant[\varepsilon(|y|)+P(|p|,|y|)](1+|p|^2), \tag{2.14}$$

where $P(|p|,|y|)\to 0$, as $|p|\to\infty$ and $\varepsilon(|y|)$ is small enough;

$$\langle h(t,x,y,p\sigma(t,x,y)),y\rangle\geqslant -L(1+|y|^2) \tag{2.15}$$

for some constant $L > 0$. *Then* (2.8) *admits a unique classical solution* $u(t,x)$ *with bounded* $u_t(t,x)$, $u_x(t,x)$, *and* $u_{xx}(t,x)$. *Moreover, there exists a constant only depending on* $\tilde{\mu}(\cdot)$, $\mu(\cdot)$, $v(\cdot)$, $\varepsilon(\cdot)$, $P(\cdot,\cdot)$, *and L appeared in* (2.11)–(2.15), *and* $||g||_\infty$, *such that*

$$|u(t,x)| + |u_x(t,x)| \leqslant C, \quad \forall (t,x) \in [0,T] \times \mathbb{R}^n. \tag{2.16}$$

Now, we introduce some more assumptions for FBSDE (1.1).

(H3) Let (2.11) hold for some continuous functions $\mu(\cdot)$ and $v(\cdot)$, such that for some constant $C > 0$,

$$0 < v(r) \leqslant \mu(r) \leqslant Cv(r)^2, \quad \forall r \geqslant 0. \tag{2.17}$$

Let also the following hold:

$$|b(t,x,0)| + |h(t,x,0,z)| \leqslant L, \quad \forall (t,x,z) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^{m\times n}. \tag{2.18}$$

(H4) Let all the functions b, σ, h are C^2 in all of their arguments, and $g \in C^{2+\alpha}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. It is not hard to check that (H2)–(H4) imply the conditions of Lemma 2.2, which leads to the unique solvability of (2.8) over $[0,T]$. Thus, of course, (2.8) admits a unique classical solution over $[s,T]$ for any $s \in [0,T)$. Hence, we have the following result.

Proposition 2.3 (Ma et al., 1994). *Let* (H1)–(H4) *hold. Then for any* $s \in [0,T)$, *the following FBSDE is uniquely solvable*:

$$\begin{aligned} &\mathrm{d}X(t) = b(t,X(t),Y(t))\,\mathrm{d}t + \sigma(t,X(t),Y(t))\,\mathrm{d}W(t), \quad t \in [s,T],\\ &\mathrm{d}Y(t) = h(t,X(t),Y(t),Z(t))\,\mathrm{d}t + Z(t)\,\mathrm{d}W(t), \quad t \in [s,T],\\ &X(s) = x, \quad Y(T) = g(X(T)). \end{aligned} \tag{2.19}$$

Moreover, the following holds:

$$\left.\begin{aligned} &Y(t) = u(t,X(t)),\\ &Z(t) = u_x(t,X(t))\sigma(t,X(t),u(t,X(t))), \end{aligned}\right\} \quad \forall t \in [s,T] \quad \text{a.s.-}\boldsymbol{P}. \tag{2.20}$$

In particular, FBSDE (1.1) *is uniquely solvable.*

In what follows, we denote the unique adapted solution of (2.19) by $(X(\cdot\,;s,x)$, $Y(\cdot,\,;s,x)$, $Z(\cdot\,;s,x))$. With this notation, (2.20) becomes

$$\left.\begin{aligned} &Y(t;s,x) = u(t,X(t;s,x)),\\ &Z(t;s,x) = u_x(t,X(t;s,x))\sigma(t,X(t;s,x),u(t,X(t;s,x))), \end{aligned}\right\} \quad t \in [s,T]. \tag{2.21}$$

Representation (2.21) can be referred to as the so-called Feynman–Kac formula (Peng, 1991).

Our goal is to establish the existence of adapted solutions to FBSDE (1.1) without assuming (H4). We point out here that (2.17) implies that

$$v(r) \geqslant \frac{1}{C} > 0, \quad \forall r \geqslant 0. \tag{2.22}$$

Thus, (H3) leads to the uniform ellipticity. We will see later that if σ is smooth, then (2.17) can be relaxed to

$$0 < \nu(r) \leqslant \mu(r), \quad \forall r \geqslant 0. \tag{2.23}$$

Then, the ellipticity is allowed to be non-uniform.

3. Approximation and convergence

We now keep (H1)–(H3). Let us introduce smooth functions b^ε: $[0,T]\times\mathbb{R}^n\times\mathbb{R}^m \to \mathbb{R}^n$, σ^ε: $[0,T]\times\mathbb{R}^n\times\mathbb{R}^m \to \mathbb{R}^{n\times n}$, h^ε: $[0,T]\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^{m\times n} \to \mathbb{R}^m$ and g^ε: $\mathbb{R}^n \to \mathbb{R}^m$ such that (with $\varphi = b, \sigma, h, g$)

$$\lim_{\varepsilon\to 0} \sup_{(t,x,y,z)\in[0,T]\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^{m\times n}} |\varphi^\varepsilon(t,x,y,z) - \varphi(t,x,y,z)| = 0. \tag{3.1}$$

Moreover, (H2) and (H3) hold for some continuous functions $\mu(\cdot), \nu(\cdot)$ satisfying (2.17) and some constant L uniform in $\varepsilon > 0$. Note that (2.11) is possible to hold for σ^ε uniformly in $\varepsilon > 0$ due to (2.17). (On the other hand, if σ itself is smooth, we do not need to approximate it and (2.17) is not needed.)

According to Proposition 2.3, the following FBSDE admits a unique adapted solution $(X^\varepsilon(\cdot), Y^\varepsilon(\cdot), Z^\varepsilon(\cdot))$:

$$\begin{aligned}
&\mathrm{d}X^\varepsilon(t) = b^\varepsilon(t, X^\varepsilon(t), Y^\varepsilon(t))\,\mathrm{d}t + \sigma^\varepsilon(t, X^\varepsilon(t), Y^\varepsilon(t))\,\mathrm{d}W(t), \quad t\in[s,T],\\
&\mathrm{d}Y^\varepsilon(t) = h^\varepsilon(t, X^\varepsilon(t), Y^\varepsilon(t), Z^\varepsilon(t))\,\mathrm{d}t + Z^\varepsilon(t)\,\mathrm{d}W(t), \quad t\in[s,T],\\
&X^\varepsilon(s) = x, \quad Y^\varepsilon(T) = g^\varepsilon(X^\varepsilon(T)).
\end{aligned} \tag{3.2}$$

Similar as above, we denote the unique adapted solution of (3.2) by $(X^\varepsilon(\cdot\,;s,x)$, $Y^\varepsilon(\cdot\,;s,x)$, $Z^\varepsilon(\cdot\,;s,x))$. Then, similar to (2.21), we have the following relations:

$$\left.\begin{aligned}
&Y^\varepsilon(t;s,x) = u^\varepsilon(t, X^\varepsilon(t;s,x)),\\
&Z^\varepsilon(t;s,x) = u^\varepsilon_x(t, X^\varepsilon(t;s,x))\sigma^\varepsilon(t, X^\varepsilon(t;s,x), u^\varepsilon(t, X^\varepsilon(t;s,x))),
\end{aligned}\right. \quad t\in[s,T], \tag{3.3}$$

where $u^\varepsilon(\cdot,\cdot)$ is the classical solution of the following:

$$\begin{aligned}
&(u^\varepsilon)^k_t + \tfrac{1}{2}\mathrm{tr}\,[(u^\varepsilon)^k_{xx}\sigma^\varepsilon(t,x,u^\varepsilon)\sigma^\varepsilon(t,x,u^\varepsilon)^{\mathrm{T}}] + \langle b^\varepsilon(t,x,u^\varepsilon), (u^\varepsilon)^k_x\rangle\\
&\qquad - (h^\varepsilon)^k(t,x,u^\varepsilon, u^\varepsilon_x\sigma^\varepsilon(t,x,u^\varepsilon)) = 0, \quad (t,x)\in[0,T)\times\mathbb{R}^n, \quad 1\leqslant k\leqslant m,\\
&u^\varepsilon(T,x) = g^\varepsilon(x), \quad x\in\mathbb{R}^n.
\end{aligned} \tag{3.4}$$

We have the following useful lemma.

Lemma 3.1. *Let* (H1)–(H3) *hold. Then the following is true*:

$$|Y^\varepsilon(t;s,x)| + |Z^\varepsilon(t;s,x)| \leqslant C, \quad \forall 0\leqslant s\leqslant t\leqslant T,\ x\in\mathbb{R}^n \quad \text{a.s.}$$

$$E\left\{\sup_{t\in[s,T]} |X^\varepsilon(t;s,x)|^m\right\} \leqslant C_m(1+|x|^m), \quad \forall (s,x)\in[0,T]\times\mathbb{R}^n,\ m\geqslant 1, \tag{3.5}$$

$$|u^\varepsilon(s_1,x) - u^\varepsilon(s_2,x)| \leqslant C|s_1 - s_2|^{1/2}, \quad \forall s_1,s_2\in[0,T],\ x\in\mathbb{R}^n. \tag{3.6}$$

Proof. First of all, applying Lemma 2.2 to (3.4), we have

$$|u^{\varepsilon}(s,x)| + |u_x^{\varepsilon}(s,x)| \leqslant C, \quad \forall (s,x) \in [0,T] \times \mathbb{R}^n, \tag{3.7}$$

with $C > 0$ independent of $\varepsilon > 0$. Thus, by (3.3), we obtain

$$|Y^{\varepsilon}(t;s,x)| \leqslant C, \quad \forall 0 \leqslant s \leqslant t \leqslant T, \quad x \in \mathbb{R}^n \quad \text{a.s.} \tag{3.8}$$

Next, by (2.11), one has

$$\begin{aligned} &|\sigma(t,X^{\varepsilon}(t;s,x),Y^{\varepsilon}(t;s,x))| \leqslant \sqrt{\mu(|Y^{\varepsilon}(t;s,x)|)} \leqslant C, \\ &\qquad \forall 0 \leqslant s \leqslant t \leqslant T, x \in \mathbb{R}^n \quad \text{a.s.} \end{aligned} \tag{3.9}$$

Consequently, combining (3.3) and (3.7) yields

$$|Z^{\varepsilon}(t;s,x)| \leqslant C, \quad \forall 0 \leqslant s \leqslant t \leqslant T, \quad x \in \mathbb{R}^n \text{ a.s.} \tag{3.10}$$

This proves the estimates for $Y^{\varepsilon}(\cdot)$ and $Z^{\varepsilon}(\cdot)$ in (3.5). Using (2.1), (2.18), (3.8)–(3.10), the Gronwall and Burkholder–Davis–Gundy inequalities, we can obtain the estimate for $X^{\varepsilon}(\cdot)$ in (3.5). Now, let us prove (3.6). Let $0 \leqslant s_1 \leqslant s_2 \leqslant T$. By (3.3), we have

$$\begin{aligned} &|u^{\varepsilon}(s_1,x) - u^{\varepsilon}(s_2,x)| \\ &\quad \leqslant E\{|u^{\varepsilon}(s_1,x) - u^{\varepsilon}(s_2,X^{\varepsilon}(s_2;s_1,x))| + |u^{\varepsilon}(s_2,X^{\varepsilon}(s_2;s_1,x)) - u^{\varepsilon}(s_2,x)|\} \\ &\quad \leqslant E|Y^{\varepsilon}(s_1;s_1,x) - Y^{\varepsilon}(s_2;s_1,x)| + CE|X^{\varepsilon}(s_2;s_1,x) - x|. \end{aligned} \tag{3.11}$$

Note that (with $X^{\varepsilon}(t) = X^{\varepsilon}(t;s_1,x)$, $Y^{\varepsilon}(t) = Y^{\varepsilon}(t;s_1,x)$, and $Z^{\varepsilon}(t) = Z^{\varepsilon}(t;s_1,x)$)

$$Y^{\varepsilon}(s_2) - Y^{\varepsilon}(s_1) = \int_{s_1}^{s_2} h^{\varepsilon}(t,X^{\varepsilon}(t),Y^{\varepsilon}(t),Z^{\varepsilon}(t))\,\mathrm{d}t + \int_{s_1}^{s_2} Z^{\varepsilon}(t)\,\mathrm{d}W(t),$$

$$X^{\varepsilon}(s_2) - x = \int_{s_1}^{s_2} b^{\varepsilon}(t,X^{\varepsilon}(t),Y^{\varepsilon}(t))\,\mathrm{d}t + \int_{s_1}^{s_2} \sigma^{\varepsilon}(t,X^{\varepsilon}(t),Y^{\varepsilon}(t))\,\mathrm{d}W(t).$$

Thus, (3.6) follows easily from (2.1), (2.18), and (3.5). □

The idea of proving (3.6) is adopted from Hu (1999b).

Now, we are ready to prove our main result of this section.

Theorem 3.2. *Let* (H1)–(H3) *hold. Then FBSDE* (1.1) *admits an adapted solution* $(X(\cdot),Y(\cdot),Z(\cdot))$.

Proof. By (3.6)–(3.7), we see that $\{u^{\varepsilon}(\cdot,\cdot)\}$ is a family of uniformly bounded and equicontinuous functions. Thus, by the Arzela–Ascoli theorem, we can extract a subsequence (we still label it by ε) such that for some continuous function $u(\cdot,\cdot)$,

$$\lim_{\varepsilon \to 0} \sup_{(t,x) \in [0,T] \times K} |u^{\varepsilon}(t,x) - u(t,x)| = 0 \tag{3.12}$$

for any compact set $K \subseteq \mathbb{R}^n$. Clearly, $u(\cdot,\cdot)$ satisfies

$$|u(s_1,x_1) - u(s_2,x_2)| \leqslant C(|s_1 - s_2|^{1/2} + |x_1 - x_2|), \quad \forall s_1,s_2 \in [0,T], \; x_1,x_2 \in \mathbb{R}^n. \tag{3.13}$$

Now, with this $u(\cdot,\cdot)$, we can solve forward SDE (2.9) to obtain the unique solution $X(\cdot)$. Next, with this $X(\cdot)$, we can solve the BSDE:

$$\begin{aligned}
&\mathrm{d}Y(t) = h(t, X(t), Y(t), Z(t))\,\mathrm{d}t + Z(t)\,\mathrm{d}W(t), \quad t \in [0, T],\\
&Y(T) = g(X(T)),
\end{aligned} \tag{3.14}$$

to get the unique adapted solution $(Y(\cdot), Z(\cdot))$. The only thing left is to prove the relation (2.3). To this end, we first apply Itô's formula to $|X^\varepsilon(\cdot) - X(\cdot)|^2$ to obtain the following:

$$\begin{aligned}
&E|X^\varepsilon(t) - X(t)|^2\\
&\quad\leqslant E\int_0^t \{2|X^\varepsilon(s) - X(s)||b^\varepsilon(s, X^\varepsilon(s), u^\varepsilon(s, X^\varepsilon(s))) - b(s, X(s), u(s, X(s)))|\\
&\qquad + |\sigma^\varepsilon(s, X^\varepsilon(s), u^\varepsilon(s, X^\varepsilon(s)) - \sigma(s, X(s), u(s, X(s)))|^2\}\,\mathrm{d}s\\
&\quad\leqslant C\int_0^t E|X^\varepsilon(s) - X(s)|^2\,\mathrm{d}s\\
&\qquad + C\int_0^t E\{|b^\varepsilon(s, X(s), u^\varepsilon(s, X(s))) - b(s, X(s), u(s, X(s)))|^2\\
&\qquad + |\sigma^\varepsilon(s, X(s), u^\varepsilon(s, X(s))) - \sigma(s, X(s), u(s, X(s)))|^2\}\,\mathrm{d}s.
\end{aligned} \tag{3.15}$$

By Gronwall's inequality, (3.1), (3.12), and the dominated convergence theorem, we obtain

$$\begin{aligned}
&E|X^\varepsilon(t) - X(t)|^2\\
&\quad\leqslant C\int_0^T E\{|b^\varepsilon(s, X(s), u^\varepsilon(s, X(s))) - b(s, X(s), u(s, X(s)))|^2\\
&\qquad + |\sigma^\varepsilon(s, X(s), u^\varepsilon(s, X(s))) - \sigma(s, X(s), u(s, X(s)))|^2\}\,\mathrm{d}s \to 0.
\end{aligned} \tag{3.16}$$

On the other hand, applying Itô's formula to $|Y^\varepsilon(\cdot) - Y(\cdot)|^2$, using (3.14) and (3.2), we have

$$\begin{aligned}
&E|g^\varepsilon(X^\varepsilon(T)) - g(X(T))|^2 - E|Y^\varepsilon(t) - Y(t)|^2\\
&\quad\geqslant E\int_t^T \{-C|Y^\varepsilon(s) - Y(s)|^2 + \frac{1}{2}|Z^\varepsilon(s) - Z(s)|^2\\
&\qquad - |h^\varepsilon(s, X^\varepsilon(s), Y(s), Z(s)) - h(s, X(s), Y(s), Z(s))|^2\}\,\mathrm{d}s.
\end{aligned} \tag{3.17}$$

Thus, by Gronwall's inequality, (3.1) and (3.16), we obtain

$$\begin{aligned}
&E|Y^\varepsilon(t) - Y(t)|^2 + E\int_0^T |Z^\varepsilon(s) - Z(s)|^2\,\mathrm{d}s\\
&\quad\leqslant C\int_0^T E|h^\varepsilon(s, X^\varepsilon(s), Y(s), Z(s)) - h(s, X(s), Y(s), Z(s))|^2\,\mathrm{d}s \to 0.
\end{aligned} \tag{3.18}$$

Consequently,

$$
\begin{aligned}
&E|Y(t)-u(t,X(t))| \\
&\quad\leqslant CE\{|Y(t)-Y^{\varepsilon}(t)|+|u^{\varepsilon}(t,X^{\varepsilon}(t))-u^{\varepsilon}(t,X(t))|+|u^{\varepsilon}(t,X(t))-u(t,X(t))|\} \\
&\quad\leqslant CE\{|Y(t)-Y^{\varepsilon}(t)|+|X^{\varepsilon}(t)-X(t)|+|u^{\varepsilon}(t,X(t))-u(t,X(t))|\}\to 0.
\end{aligned}
\tag{3.19}
$$

This shows that (1.1) admits an adapted solution $(X(\cdot),Y(\cdot),Z(\cdot))$. $\square$

Note that it is not clear if the adapted solution to (1.1) is unique.

4. One-dimensional case

In this section, we are going to discuss the one-dimensional case, by which we mean $n=m=1$. The main issue here is that we are going to allow $\sigma(t,x,y)$ to be degenerate. For technical reasons, instead of (1.1), we consider the following FBSDE:

$$
\begin{aligned}
&\mathrm{d}X(t)=b(t,X(t),Y(t))\,\mathrm{d}t+\sigma(t,X(t),Y(t))\,\mathrm{d}W(t), \quad t\in[0,T], \\
&\mathrm{d}Y(t)=[h(t,X(t),Y(t))+b(t,X(t),Y(t))Z(t)]\,\mathrm{d}t \\
&\qquad\qquad +Z(t)\sigma(t,X(t),Y(t))\,\mathrm{d}W(t), \quad \forall t\in[0,T], \\
&X(0)=x, \quad Y(T)=g(X(T)).
\end{aligned}
\tag{4.1}
$$

Note that when $\sigma(t,x,y)$ is bounded away from zero, by defining

$$
\begin{aligned}
&\bar{Z}(t)=Z(t)\sigma(t,X(t),Y(t)), \\
&\bar{h}(t,x,y,\bar{z})=h(t,x,y)+b(t,x,y)\sigma(t,x,y)^{-1}\bar{z},
\end{aligned}
\tag{4.2}
$$

we obtain an FBSDE for $(X(\cdot),Y(\cdot),\bar{Z}(\cdot))$ which is of form (1.1). We point out that form (4.1) comes naturally in problems of pricing European contingent claims in incomplete markets (see Yong, 1999b).

In what follows, we keep assumptions (H1) and (H2). We further make the following assumption (comparing with (H3)).

(H5) Let

$$
0\leqslant\sigma(t,x,y)\leqslant L, \quad \forall(t,x,y)\in[0,T]\times\mathbb{R}\times\mathbb{R} \tag{4.3}
$$

and let

$$
|b(t,x,0)|+|h(t,x,0)|\leqslant L, \quad \forall(t,x)\in[0,T]\times\mathbb{R}. \tag{4.4}
$$

Note that under (4.3), $\sigma(t,x,y)^2$ is bounded from above. The important thing is that we allow $\sigma(t,x,y)$ to be degenerate!

Our main result of this section is the following:

Theorem 4.1. *Let $n = m = 1$. Let* (H1), (H2) *and* (H5) *hold. Then FBSDE* (4.1) *admits an adapted solution* $(X(\cdot), Y(\cdot), Z(\cdot))$.

Proof. Let us introduce smooth functions $b^\varepsilon, \sigma^\varepsilon, h^\varepsilon : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, and $g^\varepsilon : \mathbb{R} \to \mathbb{R}$ such that (H2), (H4) hold and the constants L appeared in (H2) are uniform in $\varepsilon > 0$. Moreover, with $\varphi = b, \sigma, h, g$,

$$\lim_{\varepsilon \to 0} \sup_{(t,x,y) \in [0,T] \times \mathbb{R} \times \mathbb{R}} |\varphi^\varepsilon(t,x,y) - \varphi(t,x,y)| = 0. \tag{4.5}$$

In addition,

$$\varepsilon \leqslant \sigma^\varepsilon(t,x,y) \leqslant 2L, \quad \forall (t,x,y) \in [0,T] \times \mathbb{R} \times \mathbb{R}, \ \ \varepsilon > 0. \tag{4.6}$$

Note that (4.6) is possible due to (4.3). Now, consider the following FBSDE with smooth coefficients:

$$\begin{aligned} &\mathrm{d}X^\varepsilon(t) = b^\varepsilon(t, X^\varepsilon(t), Y^\varepsilon(t))\,\mathrm{d}t + \sigma^\varepsilon(t, X^\varepsilon(t), Y^\varepsilon(t))\,\mathrm{d}W(t), \quad t \in [0,T], \\ &\mathrm{d}Y^\varepsilon(t) = [h^\varepsilon(t, X^\varepsilon(t), Y^\varepsilon(t)) + b^\varepsilon(t, X^\varepsilon(t), Y^\varepsilon(t)) Z^\varepsilon(t)]\,\mathrm{d}t \\ &\qquad\qquad + Z^\varepsilon(t) \sigma^\varepsilon(t, X^\varepsilon(t), Y^\varepsilon(t))\,\mathrm{d}W(t), \quad \forall t \in [0,T], \\ &X^\varepsilon(0) = x, \quad Y^\varepsilon(T) = g^\varepsilon(X(T)). \end{aligned} \tag{4.7}$$

The same as before, suppose (4.7) admits an adapted solution $(X^\varepsilon(\cdot), Y^\varepsilon(\cdot), Z^\varepsilon(\cdot))$, which has the relation

$$Y^\varepsilon(t) = u^\varepsilon(t, X^\varepsilon(t)), \quad t \in [0,T], \tag{4.8}$$

with $u^\varepsilon(\cdot\,,\cdot)$ undetermined. By Itô's formula, we obtain the following:

$$\begin{aligned} \mathrm{d}Y^\varepsilon(t) &= \mathrm{d}u^\varepsilon(t, X^\varepsilon(t)) \\ &= \{u_t^\varepsilon(t, X^\varepsilon(t)) + u_x^\varepsilon(t, X^\varepsilon(t)) b^\varepsilon(t, X^\varepsilon(t), u^\varepsilon(t, X^\varepsilon(t))) \\ &\quad + \tfrac{1}{2} u_{xx}^\varepsilon(t, X^\varepsilon(t)) \sigma^\varepsilon(t, X^\varepsilon(t), u^\varepsilon(t, X^\varepsilon(t)))^2\}\,\mathrm{d}t \\ &\quad + u_x^\varepsilon(t, X^\varepsilon(t)) \sigma^\varepsilon(t, X^\varepsilon(t), u^\varepsilon(t, X^\varepsilon(t)))\,\mathrm{d}W(t). \end{aligned} \tag{4.9}$$

Comparing (4.9) with (4.7), we see that if u^ε is the right choice, the following should be true:

$$\begin{aligned} &h^\varepsilon(t, X^\varepsilon(t), u^\varepsilon(t, X^\varepsilon(t))) + b^\varepsilon(t, X^\varepsilon(t), u^\varepsilon(t, X^\varepsilon(t))) Z^\varepsilon(t) \\ &\quad = u_t^\varepsilon(t, X^\varepsilon(t)) + u_x^\varepsilon(t, X^\varepsilon(t)) b^\varepsilon(t, X^\varepsilon(t), u^\varepsilon(t, X^\varepsilon(t))) \\ &\qquad + \tfrac{1}{2} u_{xx}^\varepsilon(t, X^\varepsilon(t)) \sigma^\varepsilon(t, X^\varepsilon(t), u^\varepsilon(t, X^\varepsilon(t)))^2, \\ &u^\varepsilon(T, X^\varepsilon(T)) = g^\varepsilon(X^\varepsilon(T)) \end{aligned} \tag{4.10}$$

and

$$u_x^\varepsilon(t, X^\varepsilon(t)) \sigma^\varepsilon(t, X^\varepsilon(t), u^\varepsilon(t, X^\varepsilon(t))) = Z^\varepsilon(t) \sigma^\varepsilon(t, X^\varepsilon(t), u^\varepsilon(t, X^\varepsilon(t))). \tag{4.11}$$

Thus, by (4.6), it is necessary that

$$Z^{\varepsilon}(t)=u_x^{\varepsilon}(t,X^{\varepsilon}(t)), \quad t\in[0,T], \tag{4.12}$$

with $u^{\varepsilon}(\cdot,\cdot)$ being a solution of the following (compare with (3.4)):

$$\begin{aligned} &u_t^{\varepsilon}+\tfrac{1}{2}\sigma^{\varepsilon}(t,x,u^{\varepsilon})^2u_{xx}^{\varepsilon}-h^{\varepsilon}(t,x,u^{\varepsilon})=0, \quad (t,x)\in[0,T)\times\mathbb{R},\\ &u^{\varepsilon}(T,x)=g^{\varepsilon}(x), \quad x\in\mathbb{R}. \end{aligned} \tag{4.13}$$

By Lemma 2.2, we have that (4.13) admits a unique classical solution $u^{\varepsilon}(\cdot,\cdot)$ and FBSDE (4.7) admits a unique adapted solution $(X^{\varepsilon}(\cdot),Y^{\varepsilon}(\cdot),Z^{\varepsilon}(\cdot))$ with relations (4.8) and (4.12) being true. From Ladyzenskaja et al. (1967), we know that the solution $u^{\varepsilon}(\cdot,\cdot)$ of (4.13) satisfies

$$|u^{\varepsilon}(t,x)|\leqslant C, \quad \forall (t,x)\in[0,T]\times\mathbb{R},\ \ \varepsilon>0. \tag{4.14}$$

We point out here that $C>0$ is independent of $\varepsilon>0$. Next, we differentiate the equation (4.13) with respect to x to get the following (we suppress the arguments in the functions):

$$0=(u_x^{\varepsilon})_t+\tfrac{1}{2}(\sigma^{\varepsilon})^2(u_x^{\varepsilon})_{xx}+\sigma^{\varepsilon}[\sigma_x^{\varepsilon}+\sigma_y^{\varepsilon}u_x^{\varepsilon}](u_x^{\varepsilon})_x-h_x^{\varepsilon}-h_y^{\varepsilon}u_x^{\varepsilon}. \tag{4.15}$$

Thus, u_x^{ε} satisfies the following equation:

$$\begin{aligned} &(u_x^{\varepsilon})_t+\tfrac{1}{2}(\sigma^{\varepsilon})^2(u_x^{\varepsilon})_{xx}+\sigma^{\varepsilon}(\sigma_x^{\varepsilon}+\sigma_y^{\varepsilon}u_x^{\varepsilon})(u_x^{\varepsilon})_x-h_y^{\varepsilon}u_x^{\varepsilon}-h_x^{\varepsilon}=0,\\ &u_x^{\varepsilon}(T,x)=g_x^{\varepsilon}(x). \end{aligned} \tag{4.16}$$

Hence, by Ladyzenskaja et al. (1967) again, we obtain

$$|u_x^{\varepsilon}(t,x)|\leqslant C, \quad \forall (t,x)\in[0,T]\times\mathbb{R},\ \ \varepsilon>0. \tag{4.17}$$

Then the same as Lemma 3.1, we can prove that $u^{\varepsilon}(t,x)$ satisfies (3.6) as well. Consequently, there exists a subsequence (we still denote it by itself) such that (3.12) holds for some function $u(\cdot,\cdot)$ satisfying (3.13). Also, similar to the previous section, we have the following convergence:

$$\lim_{\varepsilon\to 0}E\{|X^{\varepsilon}(t)-X(t)|^2+|Y^{\varepsilon}(t)-u(t,X(t))|^2\}=0, \tag{4.18}$$

where $X(\cdot)$ is the solution of (2.9). Next, by (4.17), $Z^{\varepsilon}(\cdot)$ is bounded. Thus, we may assume that

$$Z^{\varepsilon}(\cdot)\to Z(\cdot) \quad \text{weakly in } L^2_{\mathscr{F}}(0,T;\mathbb{R}). \tag{4.19}$$

Now, we apply Itô's formula to $|Y^{\varepsilon}(t)-Y^{\delta}(t)|^2$ to obtain the following (we denote $b^{\varepsilon}(t)=b^{\varepsilon}(t,X^{\varepsilon}(t),Y^{\varepsilon}(t))$ and $b(t)=b(t,X(t),Y(t))$, etc.)

$$\begin{aligned} &E|g^{\varepsilon}(X^{\varepsilon}(T))-g^{\delta}(X^{\delta}(T))|^2-E|Y^{\varepsilon}(t)-Y^{\delta}(t)|^2\\ &\quad =E\int_t^T\{2[Y^{\varepsilon}(s)-Y^{\delta}(s)][h^{\varepsilon}(s)-h^{\delta}(s)+b^{\varepsilon}(s)Z^{\varepsilon}(s)-b^{\delta}(s)Z^{\delta}(s)]\\ &\qquad +|Z^{\varepsilon}(s)\sigma^{\varepsilon}(s)-Z^{\delta}(s)\sigma^{\delta}(s)|^2\}\,\mathrm{d}s. \end{aligned} \tag{4.20}$$

Combining the above with (3.12), (4.5) and (4.17)–(4.18), we get

$$
\begin{aligned}
&E|Y^\varepsilon(t)-Y^\delta(t)|^2+E\int_t^T|Z^\varepsilon(s)\sigma^\varepsilon(s)-Z^\delta(s)\sigma^\delta(s)|^2\,\mathrm{d}s\\
&\quad\leqslant E|g^\varepsilon(X^\varepsilon(T))-g^\delta(X^\delta(T))|^2\\
&\qquad+E\int_t^T\{2|Y^\varepsilon(s)-Y^\delta(s)|(|h^\varepsilon(s)-h^\delta(s)|+|b^\varepsilon(s)-b(s)|+|b^\delta(s)-b(s)|)\\
&\qquad+2[Y^\varepsilon(s)-Y^\delta(s)]b(s)[Z^\varepsilon(s)-Z^\delta(s)]\}\,\mathrm{d}s\to 0,\quad \varepsilon,\delta\to 0.
\end{aligned}
\tag{4.21}
$$

Thus, it is necessary that

$$
\lim_{\varepsilon\to 0}E\int_0^T|Z^\varepsilon(s)\sigma^\varepsilon(s,X^\varepsilon(s),Y^\varepsilon(s))-Z(s)\sigma(s,X(s),u(s,X(s))|^2\,\mathrm{d}s=0. \tag{4.22}
$$

Hence, by setting $Y(t)=u(t,X(t))$, we obtain an adapted solution $(X(\cdot),Y(\cdot),Z(\cdot))$ to FBSDE (4.1). □

Let us make some comments on the above result. Note that in the proof of Lemma 3.1, to obtain the estimate on $|u^\varepsilon_x(t,x)|$ (see (3.7)), we do not have to differentiate the equation (3.4) since the equation is nondegenerate. However, for the above case, Eq. (4.13) might be degenerate and we only obtain the estimate (4.14); to obtain (4.17), we have to differentiate Eq. (4.13). Since we are in the one-dimensional case, the products of scalar functions commute, thus, in differentiating (4.13), we are able to end up with (4.16). From this, we see that the arguments applied here is strictly one-dimensional. We do not know how to treat the higher dimensions at this moment.

It is interesting that our previous argument leads to the following result.

Proposition 4.2. *Let* (H2) *and* (4.3) *hold. Let*

$$
|h(t,x,0,z)|\leqslant L,\quad \forall(t,x,z)\in[0,T]\times\mathbb{R}\times\mathbb{R}. \tag{4.23}
$$

Then the following degenerate quasilinear equation admits a viscosity solution:

$$
\begin{aligned}
&u_t+\tfrac{1}{2}\sigma(t,x,u)^2u_{xx}-h(t,x,u,u_x)=0,\quad (t,x)\in[0,T)\times\mathbb{R},\\
&u(T,x)=g(x),\quad x\in\mathbb{R}.
\end{aligned}
\tag{4.24}
$$

Proof. Similar to the proof of Theorem 4.1, we have the estimates (4.14) and (4.17). Then (3.6) holds and the family $\{u^\varepsilon(\cdot,\cdot)\}$ is bounded and equicontinuous. Thus, along some subsequence, (3.12) holds. Then, noting (3.1), and by the definition and the stability of viscosity solutions (Crandall et al., 1992), we see that the limit $u(\cdot,\cdot)$ is a viscosity solution to (4.24). □

To our best knowledge, the existence of viscosity solution to equation like (4.24) seems to be new. Note that the usual monotonicity conditions stated in Crandall et al. (1992) do not hold for (4.24). Thus, it is not known if the viscosity solution is unique.

To conclude the paper, we would like to make a remark on Proposition 4.2. One sees that in (4.1), the function h is not allowed to depend on z. But (4.24) allows the function h to depend on z. In both cases, (4.14) and (4.17) hold. For (4.1), we need to obtain an adapted solution $(X(\cdot), Y(\cdot), Z(\cdot))$. Since we only have the weak convergence (4.19), if h depends on z, we are not able to pass to the limits in the approximate FBSDE. On the other hand, for (4.24), we are only interested in the viscosity solution. According to the stability of viscosity solutions (Crandall et al., 1992), the convergence of $u^{\varepsilon}(\cdot,\cdot)$ is enough and the convergence of $u_x^{\varepsilon}(\cdot,\cdot)$ is not necessary. This is why we can allow h to depend on z in (4.24).

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